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Eigensensitivity Analysis of a Defective Matrix with Zero First-Order Eigenvalue Derivatives

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A direct method is developed for calculating the first- to third-order eigenvalue derivatives and first- to second-order eigenvector derivatives of a defective matrix with a zero first-order eigenvalue derivative associated with Jordan blocks of order higher than the lowest. It is found under some conditions that the corresponding eigensolution of the perturbed problem has a distinct structure that, in the fractional power series of the perturbation parameter, the denominator of the power can not only be the orders of Jordan blocks (as in the normal cases) but also can be the sum of two successive orders of Jordan blocks. The solution method is composed of three parts. One of them is simular to known work and the other two are very different. Numerical example show the correctness of the method.

Nomenclature

$\boldsymbol{A}$	=	concerned $n \times n$ matrix
$\boldsymbol{B}$	=	$n \times n$ perturbation matrix
$d(1) < d(2) < \cdots < d(r)$	=	different orders of Jordan blocks of $\boldsymbol{A}$ associated with eigenvalue λ_0
$s(k)$	=	number of $d(k)$th-order Jordan blocks of $\boldsymbol{A}$ associated with λ_0, where $k = 1, \ldots, r$
$\boldsymbol{u}(k, l, j)$, $\boldsymbol{v}(k, j)$	=	lth-order principal vector and left eigenvector associated with jth Jordan block of order $d(k)$ related to λ_0
$\boldsymbol{w}$	=	eigenvector of $\boldsymbol{A} + \varepsilon\boldsymbol{B}$
$\boldsymbol{w}(k)$	=	kth-order perturbation coefficient of eigenvector $\boldsymbol{w}(0)$, where $k = 1, 2, \ldots$
$\boldsymbol{w}(0)$	=	differentiable eigenvector associated with λ_0
ε	=	small positive perturbation parameter
λ	=	eigenvalue of $\boldsymbol{A} + \varepsilon\boldsymbol{B}$
$\lambda(k)$	=	kth-order perturbation coefficient of eigenvalue λ_0 ($k = 1, 2, \ldots$)
λ_0	=	concerned eigenvalue of $\boldsymbol{A}$

Superscripts

H	=	transpose and complex conjugate of a matrix
T	=	transpose of a matrix

Introduction

THERE are many important applications of eigensensitivity analysis of a discrete system to dynamic analysis, identification and modification of engineering structures, and vibration control and optimization. Eigensensitivity analysis can usually be implemented in two ways. One is the modal expansion method and its extensions,[1–8] in which all or part of the eigenpairs of the unperturbed system are needed. The other is the direct method and its extensions,[9–19] in which only the eigenpairs to be differentiated are needed. Thus, it is still the most efficient method if the number of the eigenpairs to be differentiated is small compared to the size of the system.

All of the works cited concentrate on the nondefective system. Defective systems, however, can occur in practice. Persons working in the automatic control or system theory field often encounter defective systems. Defective systems can also occur in structural dynamics.[20] For example, a one-degree-of-freedom oscillator with critical damping must be a defective system, where a unique eigenvector is associated with the double eigenvalue $-\omega_0$, ω_0 being the nature undamped frequency of the oscillator. A second example is a structure in critical flutter condition, in which two couples of imaginary eigenvalues coincide and a unique couple of complex conjugate eigenvectors exists. Luongo even constructed a family of defective two-degree-of-freedom systems.[20] More important, if defective systems represent exceptional cases, nearly defective systems are more often encountered,[21] the eigensensitivity analysis of which can be done more efficiently by transferring it into that of suitable defective systems. Therefore, it is very important to know the modal sensitivities of defective systems. Because of complications and difficulties, there are very few works on the systematic study of direct methods for the eigensensitivity analysis of a defective system, and no work on modal expansion methods is found. To our knowledge, Luongo[22] is the first to analyze systematically the eigensensitivity of a defective matrix, where he gave a method that can calculate the first-order eigenvector derivatives and can be applied to the case where all of the first-order eigenvalue derivatives are distinct and nonzero. However, the eigenvalue problem of a defective system is very sensitive to the perturbations,[23] and in real situations, some of the first-order eigenvalue derivatives can be the same and zero. The case with zero first-order eigenvalue derivatives is related to the sigular perturbation situation.[21] Therefore, it is necessary to do some extensions. In Ref. 24, this work is extended to the case where some of the first-order eigenvalue derivatives are equal but nonzero. In Ref. 25 a method is given that can be used to calculate higher-order eigenvalue and eigenvector derivatives and can be applied to the case where one of the first-order eigenvalue derivative associated with the lowest-order Jordan blocks is zero. This paper extends the method to the case where one of the first-order eigenvalue derivatives associated with Jordan blocks with order higher than the lowest is zero.

Mathematical Reduction

Let

$$\tilde{\boldsymbol{A}} = \boldsymbol{A} - \lambda_0\boldsymbol{I}, \qquad \boldsymbol{V}(k) = [\boldsymbol{v}(k, 1), \ldots, \boldsymbol{v}(k, s(k))]$$

$$\boldsymbol{U}(k, 0) = \boldsymbol{0}, \qquad \boldsymbol{U}(k, l) = [\boldsymbol{u}(k, l, 1), \ldots, \boldsymbol{u}(k, l, s(k))]$$

$$l = 1, \ldots, d(k), \qquad k = 1, \ldots, r$$

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then $\boldsymbol{V}(k)$ and $\boldsymbol{U}(k,l)$ can be made to satisfy

$$\boldsymbol{V}(j)^H\tilde{\boldsymbol{A}}\boldsymbol{U}(k,l) = \begin{cases} \boldsymbol{I}_{s(k)}, & \text{when } j=k \text{ and } l=d(k) \\ \boldsymbol{0}, & \text{otherwise} \end{cases}$$

$$\tilde{\boldsymbol{A}}\boldsymbol{U}(k,l) = \boldsymbol{U}(k,l-1), \quad l=1,\dots,d(k), \qquad j,k=1,\dots,r$$

We investigate the eigenvalues and eigenvectors of $\boldsymbol{A}+\varepsilon\boldsymbol{B}$. Define

$$\boldsymbol{Q}(j,k) = \boldsymbol{V}(j)^H\boldsymbol{B}\boldsymbol{U}(k,1)$$

$$\boldsymbol{Q}(k) = \begin{bmatrix} \boldsymbol{Q}(k,k) & \cdots & \boldsymbol{Q}(k,r) \\ \cdots & \cdots & \cdots \\ \boldsymbol{Q}(r,k) & \cdots & \boldsymbol{Q}(r,r) \end{bmatrix}, \qquad \Delta(k) = \det[\boldsymbol{Q}(k)]$$

$$\boldsymbol{S}(k) = \boldsymbol{Q}(k)^{-1} = \begin{bmatrix} \boldsymbol{S}^{(k)}(k,k) & \cdots & \boldsymbol{S}^{(k)}(k,r) \\ \cdots & \cdots & \cdots \\ \boldsymbol{S}^{(k)}(r,k) & \cdots & \boldsymbol{S}^{(k)}(r,r) \end{bmatrix}$$

$$\text{when } \Delta(k) \neq 0, \qquad j,k=1,\dots,r \tag{1}$$

To investigate the variation of λ and $\boldsymbol{w}$ associated with the $s(t)$ blocks of order $d(t)(1\le t\le r)$, we usually expand λ and $\boldsymbol{w}$ in the Puiseux series of ε:

$$\lambda = \sum_{k=0}^{\infty}\lambda(k)\eta^k, \qquad \boldsymbol{w} = \sum_{k=0}^{\infty}\boldsymbol{w}(k)\eta^k \tag{2}$$

where $\eta=\varepsilon^{1/\nu}$, $\nu=d(t)$, $\lambda(0)=\lambda_0$, and $\boldsymbol{w}(0)$ is a differentiable eigenvector of $\boldsymbol{A}$, which must contain a nonzero linear combination of columns of $\boldsymbol{U}(t,1)$. When $k\ge 1$, $\lambda(k)$ and $\boldsymbol{w}(k)$ are $1/k!$ times the kth-order derivatives of the eigenvalues and eigenvectors with respect to η. In this paper, $\boldsymbol{w}(0)$ is so normalized that $w_e(0)$, the first among the components with largest absolute value, is 1: $w_e(0)=1$. Therefore, if $\boldsymbol{w}$ is so normalized that its corresponding component is 1: $w_e=1$, then when $k\ge 1$, $w_e(k)$, the corresponding component of $\boldsymbol{w}(k)$, must be 0: $w_e(k)=0$.

Substituting Eqs. (2) into

$$(\boldsymbol{A}+\varepsilon\boldsymbol{B})\boldsymbol{w} = \lambda\boldsymbol{w} \tag{3}$$

and then comparing the coefficients of the powers of η, we obtain

$$\tilde{\boldsymbol{A}}\boldsymbol{w}(j) = \sum_{k=1}^{j}\lambda(k)\boldsymbol{w}(j-k), \qquad j=0,\dots,\nu-1 \tag{4a}$$

$$\tilde{\boldsymbol{A}}\boldsymbol{w}(j) = \sum_{k=1}^{j}\lambda(k)\boldsymbol{w}(j-k) - \boldsymbol{B}\boldsymbol{w}(j-\nu), \qquad j=\nu,\nu+1,\dots \tag{4b}$$

We can formally solve Eqs. (4a) step by step and get

$$\tilde{\boldsymbol{A}}\boldsymbol{w}(0) = \boldsymbol{0} \tag{5a}$$

$$\boldsymbol{w}(0) = \sum_{k=1}^{r}\boldsymbol{U}(k,1)\boldsymbol{c}(k,0) \tag{5b}$$

$$\tilde{\boldsymbol{A}}\boldsymbol{w}(l) = \sum_{k=1}^{r}\sum_{j=1}^{l}\boldsymbol{U}(k,j)\boldsymbol{E}(l,l,k,j) \tag{5c}$$

$$\boldsymbol{w}(l) = \sum_{k=1}^{r}\sum_{j=1}^{l}\boldsymbol{U}(k,j+1)\boldsymbol{E}(l,l,k,j) + \sum_{k=1}^{r}\boldsymbol{U}(k,1)\boldsymbol{c}(k,l),\ l=1,\dots,\nu-1 \tag{5d}$$

where the definition of $\boldsymbol{E}(m,l,k,j)$ is

$$\boldsymbol{E}(m,l,k,j) = \sum_{p=j}^{m}\boldsymbol{c}(k,l-p)\sum_{\substack{h_1,\dots,h_j\ge 1\\ h_1+\cdots+h_j=p}}\prod_{q=1}^{j}\lambda(h_q)$$

and $\boldsymbol{c}(j,k)$ is an $s(k)\times 1$ to-be-determined matrix, $k=1,\dots,r$ and $j=0,1,\dots$. The condition that $\boldsymbol{w}(0)$ contains a nonzero linear combination of the columns of $\boldsymbol{U}(t,1)$ requires $\boldsymbol{c}(t,0)\neq\boldsymbol{0}$. If $\lambda(1)\neq 0$, then from the solvability conditions $\boldsymbol{V}(k)^H\tilde{\boldsymbol{A}}\boldsymbol{w}(l)=\boldsymbol{0}$, $k=1,\dots,r$ and $l=1,\dots,\nu-1$ we have

$$\boldsymbol{c}(j,k)=\boldsymbol{0}, \qquad k=0,\dots,\nu-1-d(j), \qquad j=1,\dots,t-1$$

Then $\boldsymbol{w}(l)$, $l=0,1,\dots,\nu-1$, can be expressed as

$$\boldsymbol{w}(0) = \sum_{k=t}^{r}\boldsymbol{U}(k,1)\boldsymbol{c}(k,0)$$

$$\boldsymbol{w}(l) = \sum_{k=1}^{t-1}\sum_{j=1}^{l+d(k)-\nu}\boldsymbol{U}(k,j+1)\boldsymbol{E}(l,l,k,j) + \sum_{k=t}^{r}\sum_{j=1}^{l}\boldsymbol{U}(k,j+1)\boldsymbol{E}(l,l,k,j) + \sum_{k=\min_{l+d(m)-\nu\ge 0}(m)}^{r}\boldsymbol{U}(k,1)\boldsymbol{c}(k,l), \qquad l=1,\dots,\nu-1$$

Note that when $\lambda(1)$ is solved we will have to check whether the condition $\lambda(1)\neq 0$ is satisfied. Substituting all of the preceding results into the equation for $\boldsymbol{w}(\nu)$ in Eqs. (4b), we obtain

$$\tilde{\boldsymbol{A}}\boldsymbol{w}(\nu) = \sum_{k=1}^{t-1}\lambda(1)^{d(k)}\boldsymbol{U}(k,d(k))\boldsymbol{c}(k,\nu-d(k)) + \sum_{k=t}^{r}\lambda(1)^{\nu}\boldsymbol{U}(k,\nu)\boldsymbol{c}(k,0) - \boldsymbol{B}\boldsymbol{w}(0) + \boldsymbol{G}(0) \stackrel{\text{def}}{=} \boldsymbol{R}(0) + \boldsymbol{G}(0) \tag{6}$$

The expressions of $\boldsymbol{G}(l)$ and $\tilde{\boldsymbol{G}}(l)$ in Eq. (6) and in what follows are defined by

$$\boldsymbol{G}(l) = \sum_{k=1}^{t-1}\sum_{j=1}^{d(k)-1}\boldsymbol{U}(k,j)\boldsymbol{E}(d(k)+l,\nu+l,k,j) + \sum_{k=t}^{r}\sum_{j=1}^{\nu-1}\boldsymbol{U}(k,j)\boldsymbol{E}(\nu+l,\nu+l,k,j)$$

$$\tilde{\boldsymbol{G}}(l) = \sum_{k=1}^{t-1}\sum_{j=1}^{d(k)-1}\boldsymbol{U}(k,j+1)\boldsymbol{E}(d(k)+l,\nu+l,k,j) + \sum_{k=t}^{r}\sum_{j=1}^{\nu-1}\boldsymbol{U}(k,j+1)\boldsymbol{E}(\nu+l,\nu+l,k,j)$$

On the right-hand side of Eq. (6), all of the terms except $\boldsymbol{R}(0)$ are in $\mathcal{R}(\tilde{\boldsymbol{A}})$, the range of $\tilde{\boldsymbol{A}}$. From the solvability condition of Eq. (6), $\boldsymbol{V}(j)^H\boldsymbol{R}(0)=\boldsymbol{0}$, $j=1,\dots,r$, it follows that

$$\sum_{k=t}^{r}\boldsymbol{Q}(j,k)\boldsymbol{c}(k,0) = \lambda(1)^{d(j)}\boldsymbol{c}(j,\nu-d(j)), \qquad j=1,\dots,t \tag{7a}$$

$$\sum_{k=t}^{r}\boldsymbol{Q}(j,k)\boldsymbol{c}(k,0) = \boldsymbol{0}, \qquad j=t+1,\dots,r \tag{7b}$$

In Ref. 25, we have solved the problem where $\Delta(k) \neq 0$, $k = 1, \ldots, r$, or $\Delta(1) = 0$, but $\Delta(k) \neq 0$, $k = 2, \ldots, r$. In this paper, we will investigate the problem where $\Delta(k_0) = 0$, $\Delta(k_0 - 1) \cdot \Delta(k_0 + 1) \neq 0$ (if $k_0 < r$) for some $k_0 > 1$. The eigenvalues and eigenvectors of $\boldsymbol{A} + \varepsilon \boldsymbol{B}$ associated with $s(k_0 - 1)$ Jordan blocks of order $d(k_0 - 1)$ and $s(k_0)$ blocks of order $d(k_0)$ are composed of the following three parts.

Part 1

Set $t = k_0 - 1$ and $\nu = d(t)$. Then the tth equation to the rth equation in the corresponding Eqs. (7a) and (7b) can be written in the following form of generalized eigenvalue problem:

$$[\boldsymbol{Q}(t) - \lambda(1)^{\nu} \boldsymbol{\Omega}(t)] \boldsymbol{\sigma}(t, 0) \stackrel{\text{def}}{=} \boldsymbol{B}(t) \boldsymbol{\sigma}(t, 0) = \boldsymbol{0} \tag{8}$$

where $\boldsymbol{\Omega}(t) = \mathrm{diag}(\boldsymbol{I}_{s(t)}, \boldsymbol{0}_{s(t+1)}, \ldots, \boldsymbol{0}_{s(r)})$ and $\boldsymbol{\sigma}(t, 0) = [\boldsymbol{c}(t, 0)^T, \ldots, \boldsymbol{c}(r, 0)^T]^T$. Because $\Delta(t + 1) = 0$, the determinant of $\boldsymbol{B}(t)$ is a polynomial of $\mu = \lambda(1)^{\nu}$ with order lower than $s(t)$. Because $\Delta(t) \neq 0$, any eigenvalue of problem (8) is nonzero. Thus, the condition, $\lambda(1) \neq 0$, is satisfied. In the corresponding eigenvector $\boldsymbol{\sigma}(t, 0)$, we have $\boldsymbol{c}(t, 0) \neq \boldsymbol{0}$. The solution of generalized eigenvalue problem (8) can be reduced to the solution of the following standard eigenvalue problem:

$$\boldsymbol{S}^{(t)}(t, t) \boldsymbol{c}(t, 0) = \beta \boldsymbol{c}(t, 0) \tag{9}$$

where $\beta = 1/\mu$. In this paper, it is assumed that all of the eigenvalues of problem (9) are simple. Thus, the eigenvector $\boldsymbol{c}(t, 0)$ and the left eigenvector $\tilde{\boldsymbol{c}}(t, 0)$ of problem (9) associated with same eigenvalue are not orthogonal,

$$\tilde{\boldsymbol{c}}(t, 0)^H \boldsymbol{c}(t, 0) \neq 0 \tag{10}$$

Because $\Delta(t + 1) = 0$, then $\boldsymbol{S}^{(t)}(t, t)$ must be singular so that $\beta = 0$ is an eigenvalue of problem (9), which should be discarded for our purpose. After a nonzero eigenvalue β and any corresponding eigenvector $\overset{*}{\boldsymbol{c}}(t, 0)$ and left eigenvector $\tilde{\boldsymbol{c}}(t, 0)$ of problem (9) are computed, we can calculate the corresponding eigenvector $\overset{*}{\boldsymbol{\sigma}}(t, 0) = [\overset{*}{\boldsymbol{c}}(t, 0)^T, \ldots, \overset{*}{\boldsymbol{c}}(r, 0)^T]^T$ and left eigenvector $\tilde{\boldsymbol{\sigma}}(t, 0) = [\tilde{\boldsymbol{c}}(t, 0)^T, \ldots, \tilde{\boldsymbol{c}}(r, 0)^T]^T$ of problem (8) as follows:

$$\overset{*}{\boldsymbol{c}}(k, 0) = \mu \boldsymbol{S}^{(t)}(k, t) \overset{*}{\boldsymbol{c}}(t, 0)$$

$$\tilde{\boldsymbol{c}}(k, 0) = \bar{\mu} \boldsymbol{S}^{(t)}(t, k)^H \tilde{\boldsymbol{c}}(t, 0), \qquad k = t + 1, \ldots, r$$

where the bar over a variable denotes the complex conjugate of a number. Then we compute

$$\overset{*}{\boldsymbol{w}}(0) = \sum_{k=t}^{r} \boldsymbol{U}(k, 1) \overset{*}{\boldsymbol{c}}(k, 0)$$

If the first among the components of $\overset{*}{\boldsymbol{w}}(0)$ with largest absolute value is $\overset{*}{w}_e(0)$, then we have $\boldsymbol{\sigma}(t, 0) = \overset{*}{\boldsymbol{\sigma}}(t, 0)/\overset{*}{w}_e(0)$ and $\boldsymbol{w}(0) = \overset{*}{\boldsymbol{w}}(0)/\overset{*}{w}_e(0)$. For any eigenvalue $\beta \neq 0$, we can get ν of different $\lambda(1) = \beta^{-1/\nu}$. For any $\lambda(1)$, by use of the first $t - 1$ equations in Eqs. (7a), we can compute

$$\boldsymbol{c}(j, \nu - d(j)) = \lambda(1)^{-d(j)} \sum_{k=t}^{r} \boldsymbol{Q}(j, k) \boldsymbol{c}(k, 0), \qquad j = 1, \ldots, t - 1 \tag{11}$$

Thus, $\boldsymbol{R}(0)$ can be determined, and $\tilde{\boldsymbol{w}}(\nu) = \tilde{\boldsymbol{A}}^{(1)} \boldsymbol{R}(0)$ can be calculated, where $\tilde{\boldsymbol{A}}^{(1)}$ is any generalized $\{1\}$ inverse of $\tilde{\boldsymbol{A}}$. Then $\boldsymbol{w}(\nu)$ can be expressed as

$$\boldsymbol{w}(\nu) = \tilde{\boldsymbol{w}}(\nu) + \tilde{\boldsymbol{G}}(0) + \sum_{k=1}^{r} \boldsymbol{U}(k, 1) \boldsymbol{c}(k, \nu)$$

Substituting all of the preceding results into the equations for $\boldsymbol{w}(\nu + 1)$ in Eqs. (4b), we obtain

$$\begin{aligned} \tilde{\boldsymbol{A}} \boldsymbol{w}(\nu + 1) &= \lambda(1) \tilde{\boldsymbol{w}}(\nu) + \sum_{k=1}^{t-1} \boldsymbol{U}(k, d(k)) \boldsymbol{D}(1, k, k) \\ &\quad + \sum_{k=t}^{r} \boldsymbol{U}(k, \nu) \boldsymbol{D}(1, k, t) - \boldsymbol{B} \boldsymbol{w}(1) + \boldsymbol{G}(1) \stackrel{\text{def}}{=} \boldsymbol{R}(1) + \boldsymbol{G}(1) \end{aligned} \tag{12}$$

where $\boldsymbol{D}(1, j, k)$ is defined by

$$\begin{aligned} \boldsymbol{D}(1, j, k) &= \lambda(1)^{d(k)} \boldsymbol{c}(j, \nu + 1 - d(k)) \\ &\quad + d(k) \lambda(1)^{d(k)-1} \lambda(2) \boldsymbol{c}(j, \nu - d(k)) \end{aligned}$$

When $t > 1$ and $d(t) = d(t - 1) + 1$, then $\boldsymbol{c}(t - 1, 1)$ has been calculated by Eqs. (11). Thus, the following defined quantities are known:

$$\overset{*}{\boldsymbol{w}}(1) = \begin{cases} \tilde{\boldsymbol{w}}(\nu), & \nu = 1 \\ \lambda(1) \sum_{k=t}^{r} \boldsymbol{U}(k, 2) \boldsymbol{c}(k, 0), & t = 1 \text{ and } \nu > 1, \text{ or } t > 1 \text{ and } \nu > d(t-1) + 1 \\ \boldsymbol{U}(t - 1, 1) \boldsymbol{c}(t - 1, 1) + \lambda(1) \sum_{k=t}^{r} \boldsymbol{U}(k, 2) \boldsymbol{c}(k, 0), & \text{otherwise} \end{cases}$$

$$\boldsymbol{f}(1, j) = \boldsymbol{V}(j)^H [\lambda(1) \tilde{\boldsymbol{w}}(\nu) - \boldsymbol{B} \overset{*}{\boldsymbol{w}}(1)], \qquad j = 1, \ldots, r$$

Thus, $\boldsymbol{w}(1)$ can be expressed as

$$\boldsymbol{w}(1) = \overset{*}{\boldsymbol{w}}(1) + \sum_{k=t}^{r} \boldsymbol{U}(k, 1) \boldsymbol{c}(k, 1) \tag{13}$$

All of the terms on the right-hand side of Eqs. (12) except $\boldsymbol{R}(1)$ are in $\mathcal{R}(\tilde{\boldsymbol{A}})$. From the solvability condition $\boldsymbol{V}(j)^H \boldsymbol{R}(1) = \boldsymbol{0}$, $j = 1, \ldots, r$, it follows that

$$\sum_{k=t}^{r} \boldsymbol{Q}(j, k) \boldsymbol{c}(k, 1) = \boldsymbol{D}(1, j, j) + \boldsymbol{f}(1, j), \qquad j = 1, \ldots, t \tag{14a}$$

$$\sum_{k=t}^{r} \boldsymbol{Q}(j, k) \boldsymbol{c}(k, 1) = \boldsymbol{f}(1, j), \qquad j = t + 1, \ldots, r \tag{14b}$$

Define

$$\boldsymbol{\sigma}(t, 1) = [\boldsymbol{c}(t, 1)^T, \ldots, \boldsymbol{c}(r, 1)^T]^T$$

$$\begin{aligned} \boldsymbol{F}(t, 1) &= [\nu \lambda(1)^{\nu - 1} \lambda(2) \boldsymbol{c}(t, 0)^T \\ &\quad + \boldsymbol{f}(1, t)^T, \boldsymbol{f}(1, t + 1)^T, \ldots, \boldsymbol{f}(1, r)^T]^T \end{aligned}$$

then the tth to rth equations in Eqs. (14a) and (14b) can be written as

$$\boldsymbol{B}(t) \boldsymbol{\sigma}(t, 1) = \boldsymbol{F}(t, 1) \tag{15}$$

From the solvability condition of Eqs. (15), $\tilde{\boldsymbol{\sigma}}(t, 0)^H \boldsymbol{F}(t, 1) = 0$, we obtain

$$\lambda(2) = -\frac{\sum_{k=t}^{r} \tilde{\boldsymbol{c}}(k, 0)^H \boldsymbol{f}(1, k)}{\nu \lambda(1)^{\nu - 1} \tilde{\boldsymbol{c}}(t, 0)^H \boldsymbol{c}(t, 0)}$$

When the calculated $\lambda(2)$ is substituted into Eq. (15) and any of its particular solution $\overset{*}{\boldsymbol{\sigma}}(t, 1) = [\overset{*}{\boldsymbol{c}}(t, 1)^T, \ldots, \overset{*}{\boldsymbol{c}}(r, 1)^T]^T$ is obtained, then its general solution can be expressed as

$$\boldsymbol{\sigma}(t, 1) = \overset{*}{\boldsymbol{\sigma}}(t, 1) + \alpha_1 \boldsymbol{\sigma}(t, 0)$$

where α_1 is a to be determined coefficient. Substituting the preceding expression into Eq. (13), we have

$$\boldsymbol{w}(1) = \mathring{\boldsymbol{w}}(1) + \alpha_1 \boldsymbol{w}(0) \tag{16}$$

where the known quantity $\mathring{\boldsymbol{w}}(1)$ is defined by

$$\mathring{\boldsymbol{w}}(1) = \overset{*}{\boldsymbol{w}}(1) + \sum_{k=t}^{r} \boldsymbol{U}(k,1)\overset{*}{\boldsymbol{c}}(k,1)$$

From Eq. (16) and the normalization conditions $w_e(0) = 1$ and $w_e(1) = 0$, we obtain $\alpha_1 = -\mathring{w}_e(1)$. Then by use of the first $t-1$ equations in Eqs. (14a), we can calculate

$$\boldsymbol{c}(j, \nu+1-d(j)) = \left[\sum_{k=t}^{r} \boldsymbol{Q}(j,k)\boldsymbol{c}(k,1) - d(j)\lambda(1)^{d(j)-1}\lambda(2)\boldsymbol{c}(j, \nu-d(j)) - \boldsymbol{f}(1,j) \right] \bigg/ \lambda(1)^{d(j)}$$

$$j = 1, \ldots, t-1 \tag{17}$$

Thus, $\boldsymbol{w}(1)$ and $\boldsymbol{R}(1)$ can be determined. Let $\tilde{\boldsymbol{w}}(\nu+1) = \tilde{\boldsymbol{A}}^{(1)}\boldsymbol{R}(1)$, then $\boldsymbol{w}(\nu+1)$ can be expressed as

$$\boldsymbol{w}(\nu+1) = \tilde{\boldsymbol{w}}(\nu+1) + \tilde{\boldsymbol{G}}(1) + \sum_{k=1}^{r} \boldsymbol{U}(k,1)\boldsymbol{c}(k,\nu+1) \tag{18}$$

$$\overset{*}{\boldsymbol{w}}(2) = \begin{cases} \boldsymbol{T} + \sum_{k=t-2}^{t-1} \boldsymbol{U}(k,1)\boldsymbol{c}(k,2), & t>2 \text{ and } \nu = d(t-1)+1 = d(t-2)+2 \\ \boldsymbol{T} + \boldsymbol{U}(t-1,1)\boldsymbol{c}(t-1,2), & t=2 \text{ and } \nu = d(1)+1, \text{ or } t>2 \text{ and } \nu = d(t-1)+1 > d(t-2)+2 \\ \boldsymbol{T}, & \text{otherwise} \end{cases}$$

$$\boldsymbol{f}(2,j) = \boldsymbol{V}(j)^H[\lambda(1)\tilde{\boldsymbol{w}}(\nu+1) + \lambda(2)\tilde{\boldsymbol{w}}(\nu) - \boldsymbol{B}\,\overset{*}{\boldsymbol{w}}(2)], \qquad j = 1, \ldots, r$$

then $\boldsymbol{w}(2)$ can be expressed as

$$\boldsymbol{w}(2) = \overset{*}{\boldsymbol{w}}(2) + \sum_{k=t}^{r} \boldsymbol{U}(k,1)\boldsymbol{c}(k,2) \tag{20}$$

All of the terms on the right-hand side of Eq. (19) except $\boldsymbol{R}(2)$ are in $\mathcal{R}(\tilde{\boldsymbol{A}})$. From the solvability condition $\boldsymbol{V}(j)^H\boldsymbol{R}(2) = \boldsymbol{0}$, $j = 1, \ldots, r$, it follows that

$$\sum_{k=t}^{r} \boldsymbol{Q}(j,k)\boldsymbol{c}(k,2) = \boldsymbol{D}(2,j,j) + \boldsymbol{f}(2,j), \qquad j = 1, \ldots, t \tag{21a}$$

$$\sum_{k=t}^{r} \boldsymbol{Q}(j,k)\boldsymbol{c}(k,2) = \boldsymbol{f}(2,j), \qquad j = t+1, \ldots, r \tag{21b}$$

Define

$$\boldsymbol{\sigma}(t,2) = [\boldsymbol{c}(t,2)^T, \ldots, \boldsymbol{c}(r,2)^T]^T, \qquad \boldsymbol{F}(t,2) = \begin{bmatrix} \nu\lambda(1)^{\nu-1}\lambda(2)\boldsymbol{c}(t,1) + [\nu(\nu-1)\lambda(1)^{\nu-2}\lambda(2)^2/2 + \nu\lambda(1)^{\nu-1}\lambda(3)]\boldsymbol{f}(2,t) \\ \boldsymbol{f}(2,t+1) \\ \vdots \\ \boldsymbol{f}(2,r) \end{bmatrix}$$

Substituting all of the preceding results into the equation for $\boldsymbol{w}(\nu+2)$ in Eqs. (4b), we obtain

$$\tilde{\boldsymbol{A}}\boldsymbol{w}(\nu+2) = \sum_{k=1}^{t-1} \boldsymbol{U}(k,d(k))\boldsymbol{D}(2,k,k) + \sum_{k=t}^{r} \boldsymbol{U}(k,\nu)\boldsymbol{D}(2,k,t) + \lambda(1)\tilde{\boldsymbol{w}}(\nu+1) + \lambda(2)\tilde{\boldsymbol{w}}(\nu) - \boldsymbol{B}\boldsymbol{w}(2) + \boldsymbol{G}(2) \overset{\text{def}}{=} \boldsymbol{R}(2) + \boldsymbol{G}(2) \tag{19}$$

where $\boldsymbol{D}(2,j,k)$ is defined by

$$\boldsymbol{D}(2,j,k) = \lambda(1)^{d(j)}\boldsymbol{c}(j,\nu+2-d(k)) + d(k)\lambda(1)^{d(k)-1}\lambda(2)\boldsymbol{c}(j,\nu+1-d(k)) + d(k)\Big\{\lambda(1)^{d(k)-1}\lambda(3) + \tfrac{1}{2}[d(k)-1]\lambda(1)^{d(k)-2}\lambda(2)^2\Big\}\boldsymbol{c}(j,\nu-d(k))$$

Noting the quantities computed by Eqs. (17), we see that the following defined quantities are known:

$$\boldsymbol{T} = \begin{cases} \tilde{\boldsymbol{w}}(2), & \nu = 1 \\ \tilde{\boldsymbol{w}}(2) + \sum_{k=t}^{r} \boldsymbol{U}(k,2)[\lambda(1)\boldsymbol{c}(k,1) + \lambda(2)\boldsymbol{c}(k,0)], & \nu = 2 \\ \sum_{k=t}^{r} \{\lambda(1)^2\boldsymbol{U}(k,3)\boldsymbol{c}(k,0) + \boldsymbol{U}(k,2)[\lambda(1)\boldsymbol{c}(k,1) + \lambda(2)\boldsymbol{c}(k,0)]\}, & \nu > 2 \end{cases}$$

then the tth to rth equations in Eqs. (21a) and (21b) can be written as

$$\boldsymbol{B}(t)\boldsymbol{\sigma}(t,2) = \boldsymbol{F}(t,2) \tag{22}$$

From the solvability condition of Eqs. (22), $\tilde{\boldsymbol{\sigma}}(t,0)^H\boldsymbol{F}(t,2) = 0$, we obtain

$$\lambda(3) = -\frac{1}{\nu\lambda(1)^{\nu-1}} \times \left[\frac{\nu\lambda(1)^{\nu-1}\lambda(2)\tilde{\boldsymbol{c}}(t,0)^H\boldsymbol{c}(t,1) + \sum_{k=t}^{r}\tilde{\boldsymbol{c}}(k,0)^H\boldsymbol{f}(2,k)}{\tilde{\boldsymbol{c}}(t,0)^H\boldsymbol{c}(t,0)} + \frac{1}{2}\nu(\nu-1)\lambda(1)^{\nu-2}\lambda(2)^2 \right]$$

Substitute the calculated $\lambda(3)$ into Eq. (22) and get any of its particular solutions $\overset{*}{\boldsymbol{\sigma}}(t,2) = [\overset{*}{\boldsymbol{c}}(t,2)^T, \ldots, \overset{*}{\boldsymbol{c}}(r,2)^T]^T$, and then its general solution can be expressed as

$$\boldsymbol{\sigma}(t,2) = \overset{*}{\boldsymbol{\sigma}}(t,2) + \alpha_2\boldsymbol{\sigma}(t,0)$$

where α_2 is a to-be-determined coefficient. Substituting the preceding expression into Eq. (20) we have

$$\boldsymbol{w}(2) = \mathring{\boldsymbol{w}}(2) + \alpha_2\boldsymbol{w}(0) \tag{23}$$

where the known quantity $\mathring{\boldsymbol{w}}(2)$ is defined by

$$\mathring{\boldsymbol{w}}(2) = \overset{*}{\boldsymbol{w}}(2) + \sum_{k=1}^{r} \boldsymbol{U}(k,1)\overset{*}{\boldsymbol{c}}(k,2)$$

From Eq. (23) and the normalization conditions $w_e(0) = 1$ and $w_e(2) = 0$, we obtain $\alpha_2 = -\mathring{w}_e(2)$. Then by use of the first $t-1$ equations in Eqs. (21a), we can calculate

$$\boldsymbol{c}(j, \nu + 2 - d(j)) = \lambda(1)^{-d(j)} \left\{ \sum_{k=t}^{r} \boldsymbol{Q}(j,k)\boldsymbol{c}(k,2) - d(j)\left[\lambda(1)^{d(j)-1}\lambda(3) + \frac{1}{2}(d(j)-1)\lambda(1)^{d(j)-2}\lambda(2)^2\right] \times \boldsymbol{c}(j, \nu - d(j)) - d(j)\lambda(1)^{d(j)-1}\lambda(2) \times \boldsymbol{c}(j, \nu + 1 - d(j)) - \boldsymbol{f}(2, j) \right\}, \qquad j = 1, \ldots, t-1$$

Thus, $\boldsymbol{w}(2)$ and $\boldsymbol{R}(2)$ can be determined, and $\tilde{\boldsymbol{w}}(\nu+2) = \tilde{\boldsymbol{A}}^{(1)}\boldsymbol{R}(2)$ can be calculated. The general solution of Eq. (19) can be expressed as

$$\boldsymbol{w}(\nu+2) = \tilde{\boldsymbol{w}}(\nu+2) + \tilde{\boldsymbol{G}}(2) + \sum_{k=1}^{r} \boldsymbol{U}(k,1)\boldsymbol{c}(k, \nu+2)$$

Substituting all of the preceding results into the equation for $\boldsymbol{w}(\nu+3)$ in Eqs. (4b) and using the solvability condition for the resultant equation, we can determine $\lambda(4)$ and $\boldsymbol{w}(3)$. In a similar way, the higher-order perturbation coefficients of the eigenvalues and eigenvectors can be calculated from the higher-order perturbation equations.

Part 2

Set $t = k_0$ and $\nu = d(t)$. Because $\Delta(t+1) \neq 0$, the corresponding Eqs. (7b) can be reduced to

$$\boldsymbol{c}(k,0) = -\sum_{j=t+1}^{r} \boldsymbol{S}^{(t+1)}(k,j)\boldsymbol{Q}(j,t)\boldsymbol{c}(t,0) \overset{\text{def}}{=} \boldsymbol{P}(t,k)\boldsymbol{c}(t,0), \qquad k = t+1, \ldots, r \tag{24}$$

Substituting Eqs. (24) into the tth equation in Eqs. (4a), we obtain the following standard eigenvalue problem with $\mu = \lambda(1)^\nu$ as its eigenvalue and $\boldsymbol{c}(t,0)$ as its eigenvector:

$$\left[\boldsymbol{Q}(t,t) + \sum_{k=t+1}^{r} \boldsymbol{Q}(t,k)\boldsymbol{P}(t,k)\right]\boldsymbol{c}(t,0) = \lambda(1)^\nu \boldsymbol{c}(t,0) \tag{25}$$

In this paper it is assumed that all of the eigenvalues of problem (25) are simple. Because $\Delta_t = 0$, then $\mu = 0$ is an eigenvalue of problem (25). Let $\mu_1 = 0$ and $\mu_2, \ldots, \mu_{s(t)}$ be the eigenvalues of problem (25) and $\overset{*}{\boldsymbol{c}}^{(1)}(t,0), \ldots, \overset{*}{\boldsymbol{c}}^{s(t)}(t,0)$ be the computed corresponding eigenvectors, and then compute

$$\overset{*}{\boldsymbol{c}}^{(j)}(k,0) = \boldsymbol{P}(t,k)\overset{*}{\boldsymbol{c}}^{(j)}(t,0), \qquad k = t+1, \ldots, r$$

$$\overset{*}{\boldsymbol{w}}^{(j)}(0) = \sum_{k=t}^{r} \boldsymbol{U}(k,1)\overset{*}{\boldsymbol{c}}^{(j)}(k,0), \qquad j = 1, \ldots, s(t)$$

If the first among the components of $\overset{*}{\boldsymbol{w}}^{(j)}(0)$ with largest absolute value is $\overset{*}{w}^{(j)}_{e(j)}(0)$, then we have

$$\boldsymbol{c}^{(j)}(k,0) = \frac{\overset{*}{\boldsymbol{c}}^{(j)}(k,0)}{\overset{*}{w}^{(j)}_{e(j)}(0)}, \qquad \boldsymbol{w}^{(j)}(0) = \frac{\overset{*}{\boldsymbol{w}}^{(j)}(0)}{\overset{*}{w}^{(j)}_{e(j)}(0)}, \qquad j = 1, \ldots, s(t), \qquad k = t, \ldots, r$$

Define $\lambda_1(1) = 0$ and $\lambda_k(1) = \mu_k^{1/\nu}$, $k = 2, \ldots, s(t)$, $\boldsymbol{C}(t,0) = [\boldsymbol{c}^{(1)}(t,0), \ldots, \boldsymbol{c}^{(s(t))}(t,0)]$, $\boldsymbol{W}(0) = [\boldsymbol{w}^{(1)}(0), \ldots, \boldsymbol{w}^{(s(t))}(0)]$, $\boldsymbol{\Lambda}(1) = \operatorname{diag}[\lambda_1(1), \ldots, \lambda_{s(t)}(1)]$; then by use of Eq. (25), we have

$$\left[\boldsymbol{Q}(t,t) + \sum_{k=t+1}^{r} \boldsymbol{Q}(t,k)\boldsymbol{P}(t,k)\right]\boldsymbol{C}(t,0) = \boldsymbol{C}(t,0)\Lambda(1)^\nu \tag{26}$$

Note that if $\lambda(1) = 0$ the $(t-1)$th to rth equations in Eqs. (7a) and (7b) can equivalently be written as

$$\begin{bmatrix} \boldsymbol{Q}(t-1,t-1) & \boldsymbol{Q}(t-1,t) & \cdots & \boldsymbol{Q}(t-1,r) \\ \boldsymbol{Q}(t,t-1) & \boldsymbol{Q}(t,t) & \cdots & \boldsymbol{Q}(t,r) \\ \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{Q}(r,t-1) & \boldsymbol{Q}(r,t) & \cdots & \boldsymbol{Q}(r,r) \end{bmatrix} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{c}(t,0) \\ \vdots \\ \boldsymbol{c}(r,0) \end{bmatrix} = \boldsymbol{0} \tag{27}$$

Because $\boldsymbol{c}(t,0) \neq \boldsymbol{0}$, Eqs. (27) have a nonzero solution, which contradicts the fact that $\Delta(t-1) \neq 0$. Therefore, the zero eigenvalue of problem (25) should be discarded, and the condition $\lambda(1) \neq 0$ is satisfied.

For any computed $\lambda(1) = \lambda_i(1)$, $\boldsymbol{c}(k,0) = \boldsymbol{c}^{(i)}(k,0)$, $k = t, \ldots, r$, and $\boldsymbol{w}(0) = \boldsymbol{w}^{(i)}(0)$, $2 \le i \le s(t)$, we can similarly calculate $\boldsymbol{c}(j, \nu - d(j))$, $j = 1, \ldots, t-1$, $\tilde{\boldsymbol{w}}^{(\nu)} = \tilde{\boldsymbol{A}}^{(1)}\boldsymbol{R}(0)$ as in part 1. From the solvability condition of the equation for $\boldsymbol{w}(\nu+1)$, it follows that

$$\sum_{k=t}^{r} \boldsymbol{Q}(j,k)\boldsymbol{c}(k,1) = \boldsymbol{D}(1,j,j) + \boldsymbol{f}(1,j), \qquad j = 1, \ldots, t \tag{28a}$$

$$\boldsymbol{c}(k,1) = \boldsymbol{P}(t,k)\boldsymbol{c}(t,1) + \boldsymbol{g}(1,k), \qquad k = t+1, \ldots, r \tag{28b}$$

where $\boldsymbol{D}(1,j,k)$ and the known quantities $\boldsymbol{f}(1,j)$, $j = 1, \ldots, r$, are defined as in part 1 and $\boldsymbol{g}(1,k)$, $k = t+1, \ldots, r$, are computed by

$$\begin{bmatrix} \boldsymbol{g}(1,t+1) \\ \vdots \\ \boldsymbol{g}(1,r) \end{bmatrix} = \boldsymbol{S}(t+1) \begin{bmatrix} \boldsymbol{f}(1,t+1) \\ \vdots \\ \boldsymbol{f}(1,r) \end{bmatrix}$$

Substituting Eqs. (28b) into the tth equation in Eqs. (28a) and using Eq. (26), we obtain

$$\boldsymbol{\Lambda}(1)^\nu \tilde{\boldsymbol{c}}(t,1) - \lambda(1)^\nu \tilde{\boldsymbol{c}}(t,1) = \nu\lambda(1)^{\nu-1}\lambda(2)\boldsymbol{C}(t,0)^{-1}\boldsymbol{c}(t,0) + \boldsymbol{f}(1) \tag{29}$$

where $\tilde{\boldsymbol{c}}(t,1) = \boldsymbol{C}(t,0)^{-1}\boldsymbol{c}(t,1)$ and the known quantity $\boldsymbol{f}(1)$ is computed by

$$\boldsymbol{f}(1) = \boldsymbol{C}(t,0)^{-1}\left[\boldsymbol{f}(1,t) - \sum_{k=t+1}^{r} \boldsymbol{Q}(t,k)\boldsymbol{g}(1,k)\right]$$

Noting that $\lambda(1) = \lambda_i(1)$ and $\boldsymbol{c}(t,0) = \boldsymbol{c}^{(i)}(t,0)$ and comparing the ith component on both sides of Eq. (29), we obtain

$$\lambda(2) = \lambda_i(2) = -\frac{f_i(1)}{\nu\lambda(1)^{\nu-1}}$$

Comparing the $j(\neq i)$th component on both sides of Eq. (29), we obtain the jth component of $\tilde{\boldsymbol{c}}(t,1)$,

$$\tilde{c}_j(t,1) = \frac{f_j(1)}{\lambda_j(1)^\nu - \lambda(1)^\nu}, \qquad j \neq i, \qquad j = 1, \ldots, s(t)$$

Define the known quantity

$$\boldsymbol{g}(1) = \overset{*}{\boldsymbol{w}}(1) + \sum_{k=t+1}^{r} \boldsymbol{U}(k,1)\boldsymbol{g}(1,k)$$

where the known quantity $\overset{*}{\boldsymbol{w}}(1)$ is defined as part 1, then $\boldsymbol{w}(1)$ can be expressed as

$$\boldsymbol{w}(1)=\boldsymbol{W}(0)\tilde{\boldsymbol{c}}(t,1)+\boldsymbol{g}(1)$$

Comparing the $e(=e(i))$th component on both sides of the preceding equation and using the normalized condition $w_e(0)=1$ and $w_e(1)=0$, we obtain the ith component of $\tilde{\boldsymbol{c}}(t,1)$,

$$\tilde{c}_i(t,1)=-\left[\sum_{1\le j\le s(t),j\ne i} w_e^{(j)}(0)\tilde{c}_j(t,1)+g_e(1)\right]$$

Then $\boldsymbol{c}(t,1),\boldsymbol{c}(k,1),k=t+1,\ldots,r$, and $\boldsymbol{w}(1)=\boldsymbol{w}^{(i)}(1)$ can in turn be calculated and $\boldsymbol{c}(j,\nu+1-d(j)),\ j=1,\ldots,t-1$, can similarly be computed as in part 1. Thus, $\boldsymbol{R}(1)$ can be determined, and $\tilde{\boldsymbol{w}}(\nu+1)=\tilde{\boldsymbol{A}}^{(1)}\boldsymbol{R}(1)$ can be calculated. From the solvability condition of the equation for $\boldsymbol{w}(\nu+2)$, it follows that

$$\sum_{k=t}^{r}\boldsymbol{Q}(j,k)\boldsymbol{c}(k,2)=\boldsymbol{D}(2,j,j)+\boldsymbol{f}(2,j),\qquad j=1,\ldots,t \quad (30a)$$

$$\boldsymbol{c}(k,2)=\boldsymbol{P}(t,k)\boldsymbol{c}(t,2)+\boldsymbol{g}(2,k),\qquad k=t+1,\ldots,r \quad (30b)$$

where $\boldsymbol{D}(2,j,k)$ and the known quantities $\boldsymbol{f}(2,j)(j=1,\ldots,r)$ are defined as in part 1 and $\boldsymbol{g}(2,k),\ k=t+1,\ldots,r$, are computed by

$$\begin{bmatrix}\boldsymbol{g}(2,t+1)\\ \vdots\\ \boldsymbol{g}(2,r)\end{bmatrix}=\boldsymbol{S}(t+1)\begin{bmatrix}\boldsymbol{f}(2,t+1)\\ \vdots\\ \boldsymbol{f}(2,r)\end{bmatrix}$$

Substituting Eqs. (30b) into the tth equation in Eqs. (30a) and using Eq. (26), we obtain

$$\boldsymbol{\Lambda}(1)^{\nu}\tilde{\boldsymbol{c}}(t,2)-\lambda(1)^{\nu}\tilde{\boldsymbol{c}}(t,2)=\nu\lambda(1)^{\nu-1}\lambda(3)\boldsymbol{C}(t,0)^{-1}\boldsymbol{c}(t,0)+\boldsymbol{f}(2) \quad (31)$$

where $\tilde{\boldsymbol{c}}(t,2)=\boldsymbol{C}(t,0)^{-1}\boldsymbol{c}(t,2)$ and the known quantity $\boldsymbol{f}(2)$ is computed by

$$\boldsymbol{f}(2)=\boldsymbol{C}(t,0)^{-1}\left[\boldsymbol{f}(2,t)-\sum_{k=t+1}^{r}\boldsymbol{Q}(t,k)\boldsymbol{g}(2,k)\right]+\nu\lambda(1)^{\nu-1}\lambda(2)\times\tilde{\boldsymbol{c}}(t,1)+\frac{\nu(\nu-1)}{2}\lambda(1)^{\nu-2}\lambda(2)^2\boldsymbol{C}(t,0)^{-1}\boldsymbol{c}(t,0)$$

Comparing the ith component on both sides of Eq. (31), we obtain

$$\lambda(3)=\lambda_i(3)=-\frac{f_i(2)}{\nu\lambda(1)^{\nu-1}}$$

Comparing the $j(\ne i)$th component on both sides of Eq. (31), we obtain the jth component of $\tilde{\boldsymbol{c}}(t,2)$,

$$\tilde{c}_j(t,2)=\frac{f_j(2)}{\lambda_j(1)^{\nu}-\lambda(1)^{\nu}},\qquad j\ne i,\qquad j=1,\ldots,s(t)$$

Define the known quantity

$$\boldsymbol{g}(2)=\overset{*}{\boldsymbol{w}}(2)+\sum_{k=t+1}^{r}\boldsymbol{U}(k,1)\boldsymbol{g}(2,k)$$

where the known quantity $\overset{*}{\boldsymbol{w}}(2)$ is defined as part 1; then $\boldsymbol{w}(2)$ can be expressed as

$$\boldsymbol{w}(2)=\boldsymbol{W}(0)\tilde{\boldsymbol{c}}(t,2)+\boldsymbol{g}(2)$$

Comparing the $e(=e(i))$th component on both sides of the preceding equation and using the normalized condition $w_e(0)=1$ and $w_e(2)=0$, we obtain the ith component of $\tilde{\boldsymbol{c}}(t,2)$,

$$\tilde{c}_i(t,2)=-\left[\sum_{1\le j\le s(t),j\ne i} w_e^{(j)}(0)\tilde{c}_j(t,2)+g_e(2)\right]$$

Thus, $\boldsymbol{w}(2)$ is completely determined. In a similar way we can determine $\lambda(k)$ and $\boldsymbol{w}(k-1),\ k=4,5,\ldots$, step by step.

In parts 1 and 2 we have determined $[s(k_0-1)-1]d(k_0-1)$ and $[s(k_0)-1]d(k_0)$ eigenvalues and corresponding eigenvectors of perturbed problem (3) associated with the $s(k_0-1)$ Jordan blocks of order $d(k_0-1)$ and $s(k_0)$ blocks of order $d(k_0)$. There are $d(k_0-1)+d(k_0)$ eigenvalues and the corresponding eigenvectors of problem (3) associated with the Jordan blocks mentioned earlier have not been determined yet. This is the task of part 3, which is the most complicated and difficult part.

Part 3

Set $t=k_0,\ \nu_1=d(t-1),\ \nu_2=d(t),\ \nu=\nu_1+\nu_2$, and $\eta=\varepsilon^{1/\nu}$. In this part, it is assumed that $\nu_2=\nu_1+1$. Because $\nu>d(t)$, the reduction in this part is different from that of the preceding two parts. In Eq. (5b), $\boldsymbol{w}(0)$ should contain a nonzero linear combination of the columns of $\boldsymbol{U}(t-1,1)$ and $\boldsymbol{U}(t,1)$. Thus, $\boldsymbol{c}(t-1,0)$ and $\boldsymbol{c}(t,0)$ can not be zero simultaneously. From the solvability condition of the equations for $\boldsymbol{w}(\nu_1)$ and $\boldsymbol{w}(\nu_2)$, it follows that

$$\lambda(1)^{\nu_1}\boldsymbol{c}(t-1,0)=\boldsymbol{0},\qquad \lambda(1)^{\nu_2}\boldsymbol{c}(t,0)=\boldsymbol{0}$$

Because $\boldsymbol{c}(t-1,0)$ and $\boldsymbol{c}(t,0)$ are not simultaneously zero from the preceding equations, we have $\lambda(1)=0$. Thus, if $\lambda(2)\ne 0$, then from the solvability conditions of the equations for $\boldsymbol{w}(l),\ l=1,\ldots,\nu-1$, it follows that

$$\boldsymbol{c}(j,k)=\boldsymbol{0},\qquad k=0,\ldots,\nu-1-2d(j),\qquad j=1,\ldots,t-1$$

Then $\boldsymbol{w}(l),\ l=1,\ldots,\nu-1$, can be expressed as

$$\boldsymbol{w}(0)=\sum_{k=t}^{r}\boldsymbol{U}(k,1)\boldsymbol{c}(k,0) \quad (32a)$$

$$\boldsymbol{w}(1)=\sum_{k=t-1}^{r}\boldsymbol{U}(k,1)\boldsymbol{c}(k,1) \quad (32b)$$

$$\boldsymbol{w}(2)=\lambda(2)\sum_{k=t}^{r}\boldsymbol{U}(k,2)\boldsymbol{c}(k,0)+\sum_{k=t-1}^{r}\boldsymbol{U}(k,1)\boldsymbol{c}(k,2) \quad (32c)$$

$$\boldsymbol{w}(l)=\sum_{k=1}^{t-1}\sum_{j=1}^{\left[\frac{l-\nu}{2}\right]+d(k)}\boldsymbol{U}(k,j+1)\boldsymbol{F}(l+2d(k)-\nu,l,k,j)+\sum_{k=t}^{r}\sum_{j=1}^{[l/2]}\boldsymbol{U}(k,j+1)\boldsymbol{F}(l,l,k,j)+\sum_{k=\min\limits_{2d(m)+l-\nu\ge 0}(m)}\boldsymbol{U}(k,1)\boldsymbol{c}(k,l),\qquad l=3,\ldots,\nu-1 \quad (32d)$$

where [ ] denotes the integer part of a real number and $\boldsymbol{F}(m,l,k,j)$ is defined by

$$\boldsymbol{F}(m,l,k,j)=\sum_{p=2j}^{m}\boldsymbol{c}(k,l-p)\sum_{\substack{h_1,\ldots,h_j\ge 2\\ h_1+\cdots+h_j=p}}\prod_{q=1}^{j}\lambda(h_q)$$

Note that, when $\lambda(2)$ is solved, we will have to check whether the condition $\lambda(2) \neq 0$ is satisfied. Substituting all of the preceding results into the equation for $\boldsymbol{w}(\nu)$ in Eqs. (4b), we obtain

$$\tilde{\boldsymbol{A}}\boldsymbol{w}(\nu) = \sum_{k=1}^{t-1}\sum_{j=1}^{d(k)}\boldsymbol{U}(k,j)\boldsymbol{F}(2d(k),\nu,k,j) + \sum_{k=t}^{r}\sum_{j=1}^{\nu_1}\boldsymbol{U}(k,j)\boldsymbol{F}(\nu,\nu,k,j) - \boldsymbol{B}\boldsymbol{w}(0)$$

From the solvability condition of preceding equation, it follows that

$$\sum_{k=t}^{r}\boldsymbol{Q}(j,k)\boldsymbol{c}(k,0) = \lambda(2)^{d(j)}\boldsymbol{c}(j,\nu-2d(j)), \qquad j=1,\ldots,t-1 \tag{33a}$$

$$\sum_{k=t}^{r}\boldsymbol{Q}(j,k)\boldsymbol{c}(k,0) = \boldsymbol{0}, \qquad j=t,\ldots,r \tag{33b}$$

Because $\Delta(t)=0$ the system of Eqs. (33b) has nonzero solution in which $\boldsymbol{c}(t,0)$ is the eigenvector of problem (25) associated with the zero eigenvalue. According to the assumption in part 2 that all of the eigenvalues of problem (25) are simple, the system of Eqs. (33b) only has one linearly independent nonzero solution. As in part 2, we can determine $\boldsymbol{c}(k,0)$, $k=t,\ldots,r$, and $\boldsymbol{w}(0)$ so that the first among the components of $\boldsymbol{w}(0)$ with largest absolute value is $w_e(0)=1$ and $w_e(k)=0$ for $k\geq 1$. It can be concluded that

$$\sum_{k=t}^{r}\boldsymbol{Q}(t-1,k)\boldsymbol{c}(k,0) = \lambda(2)^{\nu_1}\boldsymbol{c}(t-1,1) \neq \boldsymbol{0} \tag{34}$$

Otherwise Eq. (27) has nonzero solutions, which contradicts the fact that $\Delta(t-1)\neq 0$. Therefore, the condition $\lambda(2)\neq 0$ is satisfied. Define the known quantity

$$\tilde{\boldsymbol{w}}(\nu) = \tilde{\boldsymbol{A}}^{(1)}\left[\sum_{j=1}^{t-1}\boldsymbol{U}(j,d(j))\sum_{k=1}^{r}\boldsymbol{Q}(j,k)\boldsymbol{c}(k,0) - \boldsymbol{B}\boldsymbol{w}(0)\right]$$

then $\boldsymbol{w}(\nu)$ can be expressed as

$$\boldsymbol{w}(\nu) = \tilde{\boldsymbol{w}}(\nu) + \sum_{k=1}^{t-1}\sum_{j=1}^{d(k)-1}\boldsymbol{U}(k,j+1)\boldsymbol{F}(2d(k),\nu,k,j) + \sum_{k=t}^{r}\sum_{j=1}^{\nu_1}\boldsymbol{U}(k,j+1)\boldsymbol{F}(\nu,\nu,k,j) + \sum_{k=1}^{r}\boldsymbol{U}(k,1)\boldsymbol{c}(k,\nu)$$

Substituting all of the preceding results into the equation for $\boldsymbol{w}(\nu+1)$ in Eqs. (4b), we obtain

$$\tilde{\boldsymbol{A}}\boldsymbol{w}(\nu+1) = \sum_{k=1}^{t-1}\sum_{j=1}^{d(k)}\boldsymbol{U}(k,j)\boldsymbol{F}(2d(k)+1,\nu+1,k,j) + \sum_{k=t}^{r}\sum_{j=1}^{\nu_2}\boldsymbol{U}(k,j)\boldsymbol{F}(\nu+1,\nu+1,k,j) - \boldsymbol{B}\boldsymbol{w}(1)$$

From the solvability condition of the preceding equation, it follows that

$$\sum_{k=t-1}^{r}\boldsymbol{Q}(j,k)\boldsymbol{c}(k,1) = \lambda(2)^{d(j)}\boldsymbol{c}(j,\nu+1-2d(j)) + d(j)\lambda(2)^{d(j)-1}\lambda(3)\boldsymbol{c}(j,\nu-2d(j)), \qquad j=1,\ldots,t-1 \tag{35a}$$

$$\sum_{k=t-1}^{r}\boldsymbol{Q}(t,k)\boldsymbol{c}(k,1) = \lambda(2)^{\nu_2}\boldsymbol{c}(t,0) \tag{35b}$$

$$\sum_{k=t-1}^{r}\boldsymbol{Q}(j,k)\boldsymbol{c}(k,1) = \boldsymbol{0}, \qquad j=t+1,\ldots,r \tag{35c}$$

From Eq. (34), Eqs. (35b) and (35c) can be written as

$$\sum_{k=t}^{r}\boldsymbol{Q}(t,k)\boldsymbol{c}(k,1) = \lambda(2)^{\nu_2}\boldsymbol{c}(t,0) - \lambda(2)^{-\nu_1}\boldsymbol{Q}(t,t-1)\sum_{k=t}^{r}\boldsymbol{Q}(t-1,k)\boldsymbol{c}(k,0) \tag{36a}$$

$$\sum_{k=t}^{r}\boldsymbol{Q}(j,k)\boldsymbol{c}(k,1) = -\lambda(2)^{-\nu_1}\boldsymbol{Q}(j,t-1)\sum_{k=t}^{r}\boldsymbol{Q}(t-1,k)\boldsymbol{c}(k,0), \qquad j=t+1,\ldots,r \tag{36b}$$

Let $[\tilde{\boldsymbol{c}}(t,0)^T,\ldots,\tilde{\boldsymbol{c}}(r,0)^T]^T$ be any nonzero solution of the adjoint problem of Eqs. (33b), then we have

$$\tilde{\boldsymbol{c}}(j,0) = -\sum_{k=t+1}^{r}\boldsymbol{S}^{(t+1)}(k,j)^H\boldsymbol{Q}(t,k)^H\tilde{\boldsymbol{c}}(t,0), \qquad j=t+1,\ldots,r$$

and $\tilde{\boldsymbol{c}}(t,0)$ is a left eigenvector of problem (25) associated with its zero eigenvalue. Because zero is a simple eigenvalue of problem (25), it follows that

$$\tilde{\boldsymbol{c}}(t,0)^H\boldsymbol{c}(t,0)\neq 0$$

From the solvability condition of the system of Eqs. (36a) and (36b), we obtain

$$\lambda(2)^{\nu} = \frac{\left[\sum_{j=t}^{r}\tilde{\boldsymbol{c}}(j,0)^H\boldsymbol{Q}(j,t-1)\right]\cdot\left[\sum_{k=t}^{r}\boldsymbol{Q}(t-1,k)\boldsymbol{c}(k,0)\right]}{\tilde{\boldsymbol{c}}(t,0)^H\boldsymbol{c}(t,0)} \tag{37}$$

From Eq. (37), we can get ν different $\lambda(2)$. For any $\lambda(2)$, $\boldsymbol{c}(j,\nu-2d(j))$, $j=1,\ldots,t-1$, can be calculated from Eqs. (33a). Let $[\overset{*}{\boldsymbol{c}}(t,1)^T,\ldots,\overset{*}{\boldsymbol{c}}(r,1)^T]^T$ be any particular solution of the system of Eqs. (36a) and (36b), then

$$\boldsymbol{c}(k,1) = \overset{*}{\boldsymbol{c}}(k,1) + \alpha_1\boldsymbol{c}(k,0), \qquad k=t,\ldots,r$$

and $\boldsymbol{w}(1)$ can be expressed as

$$\boldsymbol{w}(1) = \boldsymbol{U}(t-1,1)\boldsymbol{c}(t-1,1) + \sum_{k=t}^{r}\boldsymbol{U}(k,1)\overset{*}{\boldsymbol{c}}(k,1) + \alpha_1\boldsymbol{w}(0) \overset{\text{def}}{=} \overset{*}{\boldsymbol{w}}(1) + \alpha_1\boldsymbol{w}(0)$$

where α_1 is a to-be-determined coefficient. Comparing the eth component on both sides of the preceding equation and using the normalization conditions $w_e(0)=1$ and $w_e(1)=0$, we obtain $\alpha_1 = -\overset{*}{w}_e(1)$. Thus, $\boldsymbol{w}(1)$ is determined. Define the known quantity

$$\tilde{\boldsymbol{w}}(\nu+1) = \tilde{\boldsymbol{A}}^{(1)}\left\{\sum_{j=1}^{t-1}\boldsymbol{U}(j,d(j))\sum_{k=t-1}^{r}\boldsymbol{Q}(j,k)\boldsymbol{c}(k,1) + \lambda(2)^{\nu_2}\boldsymbol{U}(t,\nu_2)\boldsymbol{c}(t,0) - \boldsymbol{B}\boldsymbol{w}(1)\right\}$$

Then from Eqs. (35a), we have

$$\boldsymbol{w}(\nu+1) = \tilde{\boldsymbol{w}}(\nu+1) + \sum_{k=1}^{t-1} \sum_{j=1}^{d(k)-1} \boldsymbol{U}(k, j+1)\boldsymbol{F}(2d(k) + 1, \nu+1, k, j) + \sum_{j=1}^{\nu_2-1} \boldsymbol{U}(t, j+1)\boldsymbol{F}(\nu+1, \nu+1, t, j) + \sum_{k=t+1}^{r} \sum_{j=1}^{\nu_2} \boldsymbol{U}(k, j+1)\boldsymbol{F}(\nu+1, \nu+1, k, j) + \sum_{k=1}^{r} \boldsymbol{U}(k, 1)\boldsymbol{c}(k, \nu+1)$$

Substituting all of the preceding results into the equation for $\boldsymbol{w}(\nu+2)$ in Eqs. (4b), we obtain

$$\tilde{\boldsymbol{A}}\boldsymbol{w}(\nu+2) = \lambda(2)\tilde{\boldsymbol{w}}(\nu) + \sum_{k=1}^{t-1} \sum_{j=1}^{d(k)} \boldsymbol{U}(k, j)\boldsymbol{F}(2d(k)+2, \nu+2, k, j) + \sum_{k=t}^{r} \sum_{j=1}^{\nu_2} \boldsymbol{U}(k, j)\boldsymbol{F}(\nu+2, \nu+2, k, j) - \boldsymbol{B}\boldsymbol{w}(2)$$

From the solvability condition of the preceding equation, it follows that

$$\sum_{k=t-1}^{r} \boldsymbol{Q}(j, k)\boldsymbol{c}(k, 2) = \boldsymbol{f}(2, j) + \lambda(2)^{d(j)}\boldsymbol{c}(j, \nu+2-2d(j)) + d(j)\lambda(2)^{d(j)-1}\lambda(3)\boldsymbol{c}(j, \nu+1-2d(j)) + d(j)\lambda(2)^{d(j)-1} \times \left\{ d(j)\lambda(2)^{d(j)-1}\lambda(4) + \frac{d(j)[d(j)-1]}{2}\lambda(2)^{d(j)-2}\lambda(3)^2 \right\} \times c(j, \nu-2d(j)), \qquad j = 1, \ldots, t-1 \tag{38a}$$

$$\sum_{k=t-1}^{r} \boldsymbol{Q}(t, k)\boldsymbol{c}(k, 2) = \boldsymbol{f}(2, t) + \lambda(2)^{\nu_2}\boldsymbol{c}(t, 1) + \nu_2\lambda(2)^{\nu_2-1}\lambda(3)\boldsymbol{c}(t, 0) \tag{38b}$$

$$\sum_{k=t-1}^{r} \boldsymbol{Q}(j, k)\boldsymbol{c}(k, 2) = \boldsymbol{f}(2, j), \qquad j = t+1, \ldots, r \tag{38c}$$

where the known quantities $\boldsymbol{f}(2, j)$, $j = 1, \ldots, r$, are defined by

$$\boldsymbol{f}(2, j) = \lambda(2)\boldsymbol{V}(j)^H \left[\tilde{\boldsymbol{w}}(\nu) - \boldsymbol{B}\sum_{k=t}^{r} \boldsymbol{U}(k, 2)\boldsymbol{c}(k, 0) \right], \qquad j = 1, \ldots, r$$

From the $(t-1)$th equation in Eqs. (35a), Eqs. (38b) and (38c) can be written as

$$\sum_{k=t}^{r} \boldsymbol{Q}(t, k)\boldsymbol{c}(k, 2) = \boldsymbol{f}(2, t) + \lambda(2)^{\nu_2}\boldsymbol{c}(t, 1) + \nu_2\lambda(2)^{\nu_2-1}\lambda(3)\boldsymbol{c}(t, 0) + \boldsymbol{Q}(t, t-1)\left[\nu_1\lambda(2)^{-1}\lambda(3)\boldsymbol{c}(t-1, 1) - \lambda(2)^{-\nu_1} \sum_{k=t-1}^{r} \boldsymbol{Q}(t-1, k)\boldsymbol{c}(k, 1) \right] \tag{39a}$$

$$\sum_{k=t}^{r} \boldsymbol{Q}(j, k)\boldsymbol{c}(k, 2) = \boldsymbol{f}(2, j) + \boldsymbol{Q}(j, t-1)\left[\nu_1\lambda(2)^{-1}\lambda(3)\boldsymbol{c}(t-1, 1) - \lambda(2)^{-\nu_1} \sum_{k=t-1}^{r} \boldsymbol{Q}(t-1, k)\boldsymbol{c}(k, 1) \right], \qquad j = t+1, \ldots, r \tag{39b}$$

From the solvability condition of the system of Eqs. (39a) and (39b) and using Eqs. (34) and (37), we obtain

$$\lambda(3) = \left[\lambda(2)^{-2\nu_1} \sum_{j=t}^{r} \tilde{\boldsymbol{c}}(j, 0)^H \boldsymbol{Q}(j, t-1) \sum_{k=t-1}^{r} \boldsymbol{Q}(t-1, k)\boldsymbol{c}(k, 1) - \lambda(2)\tilde{\boldsymbol{c}}(t, 0)^H\boldsymbol{c}(t, 1) - \lambda(2)^{-\nu_1} \sum_{j=t}^{r} \tilde{\boldsymbol{c}}(j, 0)^H \boldsymbol{f}(2, j) \right] \Bigg/ \nu\tilde{\boldsymbol{c}}(t, 0)^H\boldsymbol{c}(t, 0)$$

Thus, $\boldsymbol{c}(j, \nu+1-2d(j))$, $j = 1, \ldots, t-1$, can be calculated from Eq. (35a),

$$\boldsymbol{c}(j, \nu+1-2d(j)) = \lambda(2)^{-d(j)} \sum_{k=t-1}^{r} \boldsymbol{Q}(j, k)\boldsymbol{c}(k, 1) - d(j)\lambda(2)^{-1}\lambda(3)\boldsymbol{c}(j, \nu-2d(j)), \qquad j = 1, \ldots, t-1$$

Let $[\overset{*}{\boldsymbol{c}}(t, 2)^T, \ldots, \overset{*}{\boldsymbol{c}}(r, 2)^T]^T$ be any particular solution of the system of Eqs. (39a) and (39b), then $\boldsymbol{w}(2)$ can be expressed as

$$\boldsymbol{w}(2) = \lambda(2) \sum_{k=t}^{r} \boldsymbol{U}(k, 2)\boldsymbol{c}(k, 0) + \boldsymbol{U}(t-1, 1)\boldsymbol{c}(t-1, 2) + \sum_{k=t}^{r} \boldsymbol{U}(k, 1)\overset{*}{\boldsymbol{c}}(k, 2) + \alpha_2\boldsymbol{w}(0) \overset{\text{def}}{=} \overset{*}{\boldsymbol{w}}(2) + \alpha_2\boldsymbol{w}(0)$$

where α_2 is a to-be-determined coefficient. Comparing the eth component on both sides of the preceding equation and using the normalization conditions $w_e(0) = 1$ and $w_e(2) = 0$, we obtain $\alpha_2 = -\overset{*}{w}_e(2)$. Thus, $\boldsymbol{w}(2)$ is determined. In a similar way, we can determine $\lambda(k)$ and $\boldsymbol{w}(k-1)$, $k = 4, 5, \ldots,$ step by step.

Numerical Example

Consider the 12×12 matrix $\boldsymbol{A}$ in Ref. 24 and the 12×12 matrix

$$\boldsymbol{B} = \begin{bmatrix} 2 & -1 & 0 & -1 & 2 & -2 & 2 & -1 & -1 & 2 & -1 & 0 \\ 4 & -2 & 0 & -2 & 4 & -4 & 4 & -2 & -2 & 4 & -2 & 0 \\ 6 & -3 & 0 & -2 & 4 & -5 & 6 & -3 & -3 & 6 & -3 & 0 \\ 8 & -4 & 0 & -2 & 4 & -6 & 8 & -4 & -4 & 8 & -4 & 0 \\ 10 & -5 & 0 & -2 & 4 & -7 & 10 & -5 & -4 & 8 & -4 & 0 \\ 12 & -6 & 0 & -2 & 4 & -8 & 12 & -6 & -4 & 8 & -4 & 0 \\ 14 & -6 & -2 & -1 & 4 & -9 & 14 & -7 & -4 & 8 & -4 & 0 \\ 16 & -6 & -4 & 0 & 4 & -10 & 16 & -8 & -4 & 8 & -4 & 0 \\ 18 & -6 & -6 & 1 & 4 & -11 & 18 & -9 & -4 & 8 & -4 & 0 \\ 20 & -6 & -8 & 2 & 4 & -11 & 18 & -9 & -4 & 8 & -4 & 0 \\ 22 & -6 & -10 & 3 & 4 & -11 & 18 & -9 & -4 & 8 & -4 & 0 \\ 24 & -6 & -12 & 4 & 4 & -11 & 18 & -9 & -4 & 8 & -4 & 0 \end{bmatrix}$$

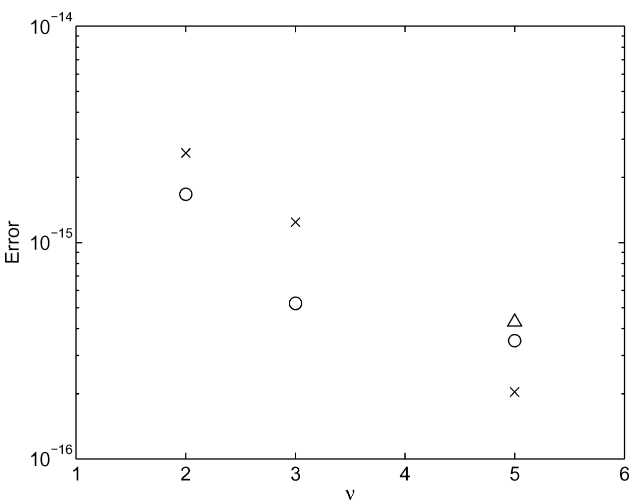
All of the eigenvalues of $\boldsymbol{A}$ are 1. In the Jordan canonical form of $\boldsymbol{A}$, there are three blocks of order two and two blocks of order three.

We can take $\boldsymbol{U}(1, 1)$, $\boldsymbol{U}(1, 2)$, $\boldsymbol{U}(2, 1)$, $\boldsymbol{U}(2, 2)$, $\boldsymbol{U}(2, 3)$, $\boldsymbol{V}(1)$, and $\boldsymbol{V}(2)$ as in Ref. 24. It is easy to see that $\Delta(2) = 0$ but $\Delta(1) \neq 0$. The characteristic equation of $\boldsymbol{A} + \varepsilon\boldsymbol{B}$ is

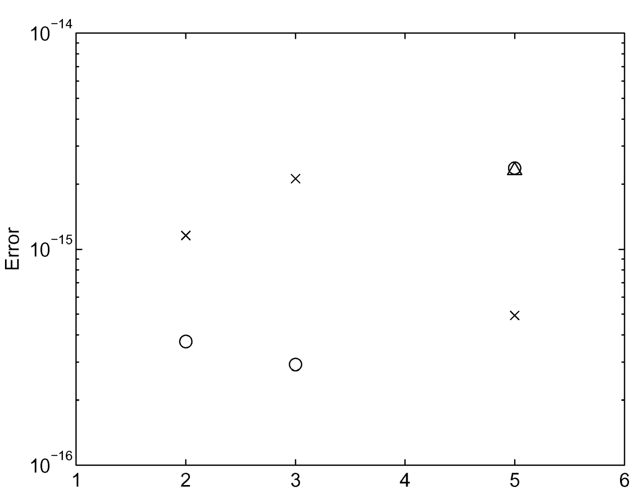
$$\hat{\lambda}^{12} - \varepsilon\hat{\lambda}^9 + \varepsilon^2\hat{\lambda}^7 - \varepsilon^3\hat{\lambda}^4 - \varepsilon^4\hat{\lambda}^3 + \varepsilon^5 = (\hat{\lambda}^3 - \varepsilon)(\hat{\lambda}^9 + \varepsilon^2\hat{\lambda}^4 - \varepsilon^4) = 0 \qquad (40)$$

where $\hat{\lambda} = \lambda - 1$. Three of the eigenvalues of $\boldsymbol{A} + \varepsilon\boldsymbol{B}$ are $1 + \varepsilon^{1/3}$ and $1 + \varepsilon^{1/3}[-\frac{1}{2} \pm (\sqrt{3}/2)i]$. We can not get the exact solution of the other eigenvalues of $\boldsymbol{A} + \varepsilon\boldsymbol{B}$. By direct substitution of $\eta = \varepsilon^{1/\nu}$ for $\nu = 1, \ldots, 12$ into Eq. (40) and comparing the powers of η, we can find that only $\nu = 2$, 3, and 5 are valid and that, for $\nu = 2$ and 3, the first- to third-order perturbation coefficients of the eigenvalues are, respectively, $1, -\frac{1}{4}, 15/32$; $-1, -\frac{1}{4}, -15/32$; $i, \frac{1}{4}, -(15/32)i$; and $-i, \frac{1}{4}, (15/32)i$ and $1, 0, 0$; $-\frac{1}{2} + (\sqrt{3}/2)i, 0, 0$; and $-\frac{1}{2} - (\sqrt{3}/2)i, 0, 0$. For $\nu = 5$, the first- to fourth-order perturbation coefficients of the eigenvalues are $0, -1, 0, 0.2$; $0, \exp(\pi i/5), 0, 0.2\exp(2\pi i/5)$; $0, \exp(-\pi i/5), 0, 0.2\exp(-2\pi i/5)$; $0, \exp(3\pi i/5), 0, 0.2\exp(6\pi i/5)$; and $0, \exp(-3\pi i/5), 0, 0.2\exp(-6\pi i/5)$.

The calculations were performed with 15 significant decimal digits. The generalized {1} inverse of singular matrices were computed by Gaussian elimination with complete pivot. All of the calculated first- to third-order perturbation coefficients of the eigenvalues are correct almost within machine accuracy. Substituting all of the computed differentiable eigenvectors, the first- to second-order eigenvector derivatives and the computed eigenvalue derivatives mentioned earlier into the equations for $\boldsymbol{w}(1)$ and $\boldsymbol{w}(2)$ in Eqs. (4b), we can get the errors for these equations. Figure 1 shows the l_1 norm errors of the equation for $\boldsymbol{w}(1)$ and $\boldsymbol{w}(2)$. It can be seen from Fig. 1 that all of the equations are satisfied almost within machine accuracy.



a)



b)

Fig. 1 Errors of equations for a) $w(1)$ and b) $w(2)$: for $\nu = 2$; $\times$, $\lambda(1) = \pm 1$ and $\circ$, $\lambda(1) = \pm i$; for $\nu = 3$, $\times$, $\lambda(1) = 1$ and $\circ$, $\lambda(1) = -\frac{1}{2} \pm (\sqrt{(3)}/2)i$; and for $\nu = 5$, $\times$, $\lambda(2) = -1$, $\circ$, $\lambda(2) = \exp(\pm\pi i/5)$; and $\triangle$, $\lambda(2) = \exp(\pm 3\pi i/5)$.

Conclusions

In the case where $\Delta(k_0) = 0$ for some $k_0 > 1$ but $\Delta(k_0 - 1) \neq 0$ and $\Delta(k_0 + 1) \neq 0$, we give a direct method to calculate the first- to third-order perturbation coefficients of the eigenvalues and first- to second-order perturbation coefficients of the eigenvectors of a defective eigenvalue λ_0 associated with the $d(k_0 - 1)$th- and $d(k_0)$th-order Jordan blocks under the conditions that all of the eigenvalues of problems (9) and (25) are simple and that $d(k_0) = d(k_0 - 1) + 1$. A numerical example shows the validity of the method. When $d(k_0) \neq d(k_0 - 1) + 1$, the η in part 3 could be $\varepsilon^{1/\nu}$ for some ν satisfying $d(k_0) < \nu < d(k_0) + d(k_0 - 1)$, and the method in part 3 can not be used. In this case, as well as in cases where $\Delta(k_0) = 0$ and $\Delta(k_0 - 1) = 0$ for some $k_0 > 1$, we must find some new way to do higher-order eigensensitivity analysis.

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